# *Non-Uniform Convergence and the Integration of Series Term by Term.*

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The subject of this paper is the study (I) of the manner of the convergence of a function $s_n(x)$, satisfying Conditions $(A)$, when $n$ becomes infinite (v. §§1, 2); and (II) of the conditions under which

$$\int_{x_0}^{x} \lim_{n=\infty} s_n(x)\,dx = \lim_{n=\infty} \int_{x_0}^{x} s_n(x)\,dx$$

(v. §13). The principal results are stated in the italicized theorems and paragraphs, and a table of contents is added, chiefly to enable the reader to inform himself more readily concerning the nomenclature.

The four problems of

1) integration of a series term by term,
2) differentiation of a series term by term,
3) reversal of the order of integration in a double-integral,
4) differentiation under the sign of integration,

are in certain classes of cases but different forms of the same problem, a problem in double limits; so that a theorem applying to one of these problems yields at once a theorem applying to the other three. It did not seem best to go into this question in this paper, but one such theorem is stated as an example in the last paragraph.

## I.—Non-Uniform Convergence.

1. 1) Let $u_1(x)$, $u_2(x)$, .... be a set of single-valued real functions of the real variable $x$, which for all values of $x$ in the finite interval $L: a \leqq x \leqq b$ are continuous. Form the sum:

$$s_n(x) = u_1(x) + u_2(x) + \ldots. + u_n(x),$$

then $s_n(x)$ is also continuous in $L$.*

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* In this paper an interval is to be understood as including among its points its extremities.

2) If a point $x$ of $L$ be chosen at pleasure and then regarded as fixed, $s_n(x)$ shall converge toward a limit when $n$ increases indefinitely. Denote this limit by $f(x)$:

$$f(x) = \lim_{n=\infty} s_n(x) = u_1(x) + u_2(x) + \dots.$$

3) Finally, $f(x)$ shall be a continuous function of $x$.

*Conditions* $(A)$.—This set of conditions will be referred to in the following pages as *Conditions* $(A)$.

We propose to investigate the most general manner of the convergence of $s_n(x)$ toward its limit $f(x)$ as $n$ increases indefinitely, *when no further restrictions are placed on* $s_n(x)$ *than Conditions* $(A)$.

2. In general the convergence will be non-uniform. Examples of such convergence are now familiar in *arithmetic* form. Thus:

*Ex.* 1. $$s_n(x) = \frac{n^2x}{1+n^3x^2}, \qquad f(x) = 0, \qquad 0 \leqq x \leqq 1.$$

The geometric representation of the non-uniform convergence by means of the approximation curves

$$y = s_n(x)$$

is given in Fig. 1. The peaks rise higher and higher as $n$ increases and their

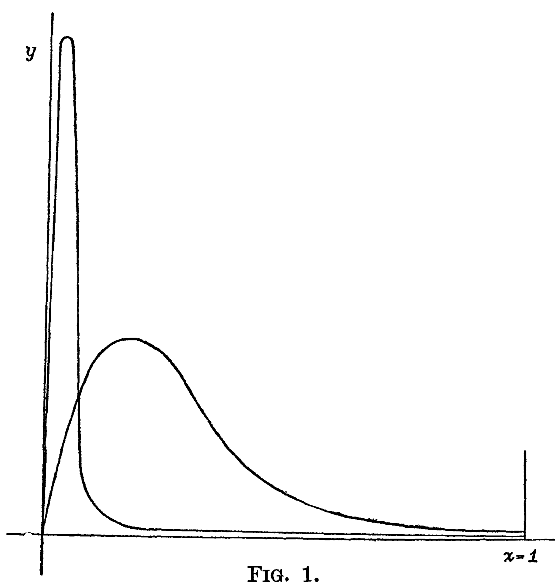


FIG. 1.

altitude transcends ultimately any magnitude $m$ chosen at pleasure, no matter

how large. I will express this in the form: $s_n(x)$ has infinite peaks in the neighborhood of the point $x = 0$.

*Definition of X-Points.*—A point $x_0$ in whose neighborhood $s_n(x)$ has infinite peaks shall be denoted by $X$ and designated as a $X$-point.

*Ex.* 2. $$s_n(x) = \frac{nx}{1 + n^2x^2}, \qquad f(x) = 0, \qquad 0 \leqq x \leqq 1.$$

Here the altitude of each peak is $\frac{1}{2}$ (Fig. 2).

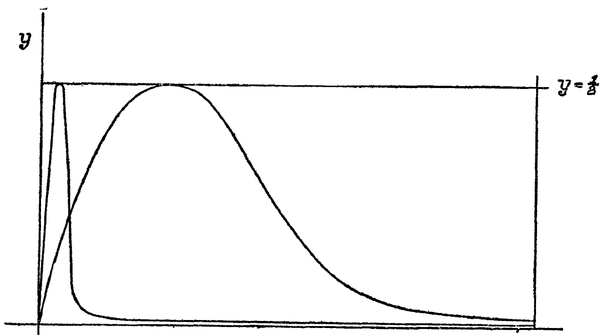


FIG. 2.

In the above examples there is only *one* point in the interval $0 \leqq x \leqq 1$ in whose neighborhood peaks occur, namely the point $x = 0$. And if any interval not abutting on the point $x = 0$ be picked out from the above interval, $s_n(x)$ will, in it, converge uniformly toward its limit. But it is easy to see that such points can occur frequently. Thus if in either of the above examples $x$ be replaced by $\sin^2 k\pi x$, where $k$ is a constant positive integer, peaks will occur in the neighborhood of each of the $(k + 1)$ points (Fig. 3),

$$0, \ 1/k, \ 2/k, \ldots . \ k/k = 1.$$

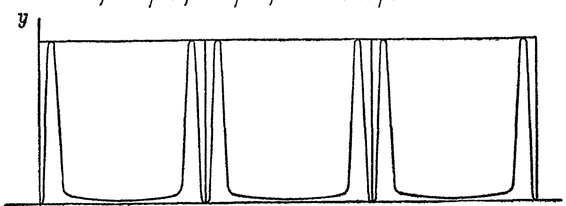


FIG. 3.

*Ex.* 3. In fact, by the aid of Ex. 2, a function $s_n(x)$ can be readily formed such that, if $x_0$ be any rational value of $x$ whatsoever, peaks will occur in the neighborhood of $x_0$. Let

$$\phi_k(x) = \frac{n \sin^2 k\pi x}{1 + n^2 \sin^4 k\pi x}$$

and form the series:

$$s_n(x) = \phi_{1!}(x) + \frac{1}{2!}\phi_{2!}(x) + \frac{1}{3!}\phi_{3!}(x) + \ldots\ldots = \sum_{i=1}^{\infty}\frac{1}{i!}\phi_{i!}(x).$$

This series, regarded as a function of the two independent variables $n, x$, converges uniformly. Hence for a given value of $n$, since each of the terms is a continuous function of $x$, the series defines a continuous function of $x$; while for a given value of $x$

$$\lim_{n=\infty} s_n(x) = \lim_{n=\infty}\phi_{1!}(x) + \lim_{n=\infty}\frac{1}{2!}\phi_{2!}(x) + \ldots\ldots = 0.$$

It is easy to see how the approximation curves

$$y = s_n(x)$$

look. Think of $n$ as large. Then the earlier terms in the $s_n(x)$ series will, individually, appear as indicated in Fig. 4 (these curves are all similar to each

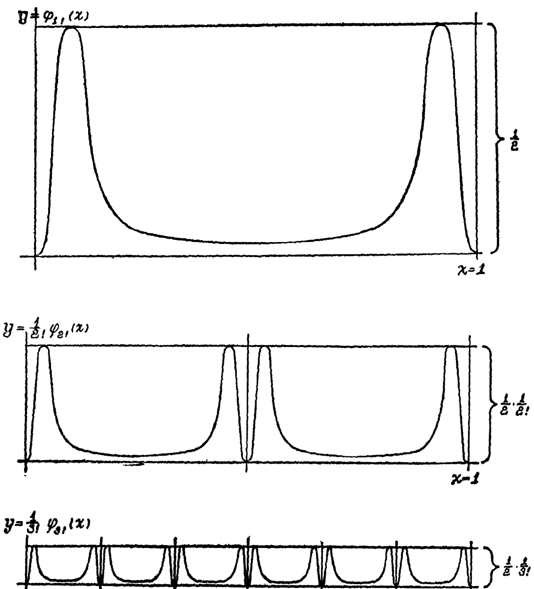


FIG. 4.

other), while the later terms, their maximum ordinates being very small, yield but a small contribution to the sum. Thus in the neighborhood of $x = \frac{1}{2}$ the peaks that come from $\frac{1}{2!}\phi_{2!}(x)$, their height being $\frac{1}{2}\cdot\frac{1}{2!}$, are predominant in

determining the character of $s_n(x)$ in the neighborhood of this point; for all the subsequent terms together cannot for any value of $x$ yield a sum greater than

$$\frac{1}{2}\cdot\frac{1}{3!}+\frac{1}{2}\cdot\frac{1}{4!}+\ldots<\frac{1}{2}\;\frac{1}{3!}\;\frac{4}{3}<\frac{1}{2}\cdot\frac{1}{4}$$

or less than 0, while the contribution of the preceding terms $(\phi_{1!}(x))$ is small in the neighborhood of $x=\frac{1}{2}$. And so it is in the general case. Let $x=\frac{p}{q}$ be any positive fraction less than unity, in reduced form. Let $k$ be the smallest integer such that $k!$ is divisible by $q$. Then in the neighborhood of the point $x=\frac{p}{q}$, the peaks that come from the term $\frac{1}{k!}\phi_{k!}(x)$ will be predominant in determining the character of $s_n(x)$ in the neighborhood of this point. Hence this function converges non-uniformly toward its limit *in every interval** that can be chosen from the given interval $0\leqq x\leqq 1$. The upper limit (resp. maximum) of the heights of the peaks is not, however, the same in all intervals; in fact, in some intervals it is very small. We may think of this upper limit as measuring, so to speak, the *strength* of the non-uniform convergence and say: the non-uniformity of the convergence is stronger in some intervals than in others.

3. We turn now to the general case and prove a fundamental theorem concerning the manner of the convergence of $s_n(x)$ toward its limit.

That which is essential in this matter will come out more clearly if we study, not the function $s_n(x)$, but

$$S_n(x)=s_n(x)-f(x),$$

the limit of which for a given value of $x$ and $n=\infty$ is $0$. This substitution amounts arithmetically to subtracting $f(x)$ from the first term of the series, $u_1(x)$; geometrically, to dropping the curve $f(x)$ down on the $x$-axis, as indicated

*The method by which this function has been formed is essentially the same as Hankel's Princip der Verdichtung der Singularitäten. Cf. Hankel, Unendlich oft oscillirende und unstetige Functionen, Tübingen, 1870; Dini, Fondamenti per la teorica delle funzioni di variabili reali, Pisa, 1878; German translation by Lüroth and Schepp, Leipzig, 1892.

in Fig. 5, the approximation curves $y = s_n(x)$ having each of their ordinates

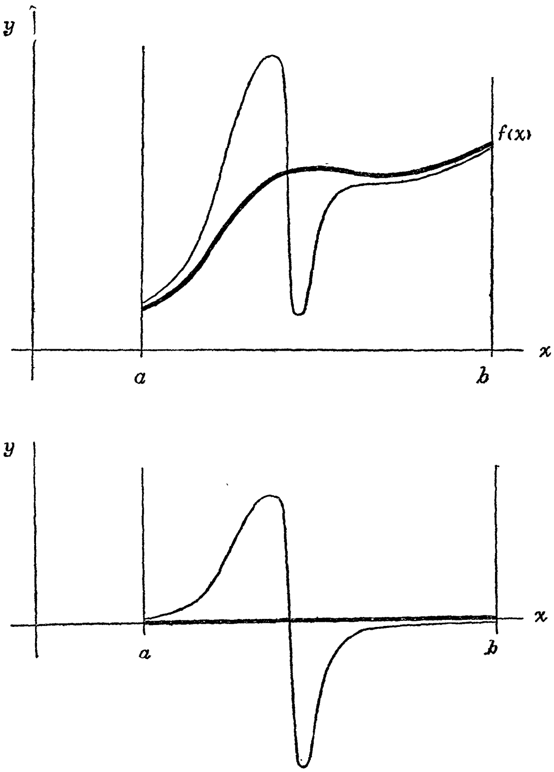


FIG. 5.

changed by the same amount. $S_n(x)$, like $s_n(x)$, satisfies Conditions ($A$) and further,

$$\lim_{n=\infty} S_n(x) = 0.$$

Let $A$ be a positive quantity chosen at pleasure and let $x_0$ be any point of $L$ (Fig. 6). Since $\lim\limits_{n=\infty} S_n(x_0) = 0$, there exists a fixed integer $m$ such that

$$|S_n(x_0)| < A, \qquad n \geqq m.$$

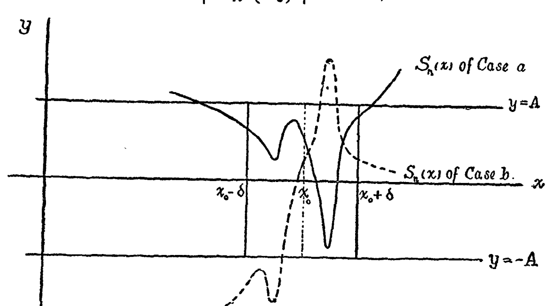


FIG. 6.

Consider the behavior of $S_n(x)$ in the neighborhood of $x_0$ when $n$ increases indefinitely. One of two cases must arise; either $a$) it is possible to name a positive quantity $\delta$ and a positive integer $m$ such that for all values of $x$ lying in the interval $(x_0 - \delta,\ x_0 + \delta)$

$$|S_n(x)| < A, \qquad n \geqq m;$$

or $b$) no matter how small $\delta$ and how large $m$ be taken (and then held fast), a value of $n$ greater than $m$ can then be chosen so that

$$|S_n(x)| > A$$

somewhere in the interval $(x_0 - \delta,\ x_0 + \delta)$.

*Definition of $\gamma$-Points.* In Case $b$) $x_0$ shall be denoted by $\gamma$ and designated as a $\gamma$-point.

Such points depend in general on the choice of $A$. If a new value $A' < A$ be taken, all the points $\gamma_A$ that formerly were $\gamma$-points remain such, but new points may claim membership in the $\gamma_{A'}$-list.

*Conditions* $(P)$. A set of points* which 1) is nowhere dense and 2) contains its derivative shall be said to satisfy *Conditions* $(P)$. Such a set will usually be denoted by the letter $G$, to which suffixes may be attached. An example typical for the most general set $G$ is given in §8.

FUNDAMENTAL THEOREM.—*Let the positive quantity $A$ be chosen at pleasure. Then the corresponding $\gamma$-points form a set of points $G$ satisfying Conditions* $(P)$.

The proof is as follows. That $G$ contains its derivative is evident from the nature of the $\gamma$-points. Suppose that throughout a certain sub-interval $(\alpha,\ \beta)$ of

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* No technical knowledge of the theory of Sets of Points (Punktmengen) will be assumed in this paper (except in §21, where explicit references are given). But the conceptions involved in the definitions to be given presently will be necessary for an understanding of what follows. Cf. G. Cantor's early papers in Crelle and the Math. Ann., translated into French and reprinted in the Acta Math., vol. 2, 1883; also Dini, Ch. II.

I will translate *Punktmenge* = *ensemble* by *set of points; überall dicht* = *condensé* by *dense; Ableitung* = *dérivé* by *derivative; abzählbar* = *dénombrable* by *enumerable.*

A set of points is said to be *dense throughout an interval* if, $x_0$ being any point lying within the interval, it is impossible to enclose $x_0$ within an interval $(x_0 - \delta,\ x_0 + \delta)$ which is free from points of the set. If an interval $L$ contains no sub-interval whatsoever throughout which the given set is dense, then the set is said to be *nowhere dense in L*. By the *derivative* of a set of points is meant that set whose points are each the limit approached by some sub-set of points of the original set. Thus each point of the derivative is a *limiting point* (*Häufungsstelle*) of the original set. A set of points is said to *contain its derivative* if the points of the derivative set all belong to the given set.

$L$ the $\gamma$-points were dense (Fig. 7). Let $c$ be any $\gamma$-point of this interval except-

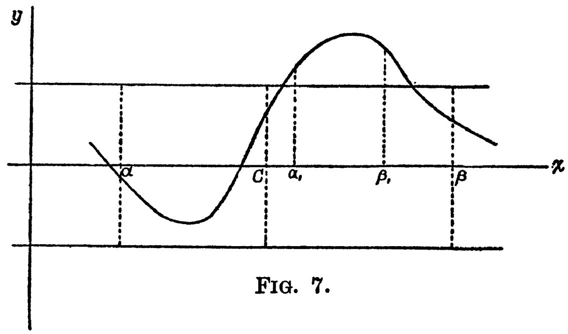


FIG. 7.

ing an extremity: $\alpha < c < \beta$. Then $n_1$ can be so determined that in portions of at least one of the intervals $(\alpha, c)$, $(c, \beta)$

$$|S_{n_1}(x)| > A.$$

It will thus be possible to separate from $(\alpha, \beta)$ a first sub-interval $(\alpha_1, \beta_1)$ such that

$$|S_{n_1}(x)| > A, \qquad \alpha_1 \leqq x \leqq \beta_1.$$

Next proceed with the sub-interval $(\alpha_1, \beta_1)$ in the same way as originally $(\alpha, \beta)$ was treated. In this way an $S_{n_2}(x)$, $n_2 > n_1$, and a sub-interval $(\alpha_2, \beta_2)$ are found such that $\alpha_1 \leqq \alpha_2$, $\beta_2 \leqq \beta_1$ and

$$|S_{n_2}(x)| > A, \qquad \alpha_2 \leqq x \leqq \beta_2.$$

The repetition of this step leads to a series of functions $S_{n_i}(x)$ and a series of intervals $(\alpha_i, \beta_i)$, where $\alpha_i < \beta_i$, $\alpha \leqq \alpha_1 \leqq \alpha_2 \ldots .$, $\beta \geqq \beta_1 \geqq \beta_2 \ldots .$, and

$$|S_{n_i}(x)| > A, \qquad \alpha_i \leqq x \leqq \beta_i.$$

Thus the $\alpha$'s and the $\beta$'s converge toward limits, and they shall be so chosen that these limits are equal:

$$\lim_{i=\infty} \alpha_i = \lim_{i=\infty} \beta_i = \bar{c}.$$

Then is $$\alpha_i < \bar{c} < \beta_i.$$

Hence $$|S_{n_j}(\bar{c})| > A, \qquad j = 1, 2, \ldots .$$

But this is impossible, since $\lim\limits_{n=\infty} S_n(\bar{c}) = 0$, and thus the theorem is established.

4. By the aid of the Fundamental Theorem we will now study the manner of the convergence of $s_n(x)$ toward its limit $f(x)$.

THEOREM I.—*The $X$-points of the interval $L$ form* AT MOST *a set $G$ satisfying Conditions $(P)$.**

From the nature of the $X$-points it is evident that $G$ contains its derivative. Furthermore these $X$-points will also be $X$-points of $S_n(x)$. Let the positive quantity $A$ be taken at pleasure. The corresponding $\gamma_A$-points will by the Fundamental Theorem form a set satisfying Conditions $(P)$ and will in general be composed of two classes of $\gamma$-points: $a$) those $\gamma$-points which drop out (cease to be $\gamma$-points) when the value of $A$ is increased; and $b$) other points that remain $\gamma$-points, no matter how large $A$ be taken. These latter are the $X$-points, and since they are at most a part of a set nowhere dense, they must themselves form such a set. Moreover this set contains its derivative. The $X$-points form therefore a set satisfying Conditions $(P)$.

5. It thus appears that if $x_0$ is any point of the interval $L$, there will be in every neighborhood of $x_0$ an interval which is entirely free from $X$-points. To such an interval $(\alpha, \beta)$ there corresponds a fixed number $B$, different for different intervals, such that

$$|S_n(x)| < B, \qquad \alpha \leqq x \leqq \beta, \qquad n = 1, 2, \ldots .$$

For suppose this were not the case. Then there would exist in $(\alpha, \beta)$ an $x_1$ for

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* Du Bois Reymond has given in the Sitz.-Ber. d. Berliner Akad., 1886, p. 359, an example of a function which satisfies, as he believed, Conditions (A), while its $X$-points are dense throughout an interval, and hence from their nature fill the interval. His proof turns ultimately on the limits approached by certain complicated expressions, and at this part of his paper he restricts himself to assertions (top of p. 370). But at an earlier point of the paper (p. 363, line 15) he says, referring to Kronecker's sufficient condition that a series may be integrated term by term (ibid. 1878, p. 54): "Aber diese ausreichende Bedingung ist offenbar (sic) notwendig, mithin ist sie mit der Forderung $Q(x) = 0$ [in the notation of this paper, $Q(x) = \lim_{n=\infty} \int_{x_0}^{x_1} S_n(x)\,dx$] vollständig aequivalent." From this and subsequent passages it appears that he overlooked the possibility of the $X$-points resp. the $\gamma$-points forming the most general set of points satisfying Conditions (P).

which

$$|S_{n_1}(x_1)| > 1;$$

an $x_2$ for which

$$|S_{n_2}(x_2)| > 2, \qquad n_2 > n_1,$$

etc.; generally, an $x_i$ for which

$$|S_{n_i}(x_i)| > i, \qquad n_i > n_{i-1}.$$

The set of points $x_1, x_2, \ldots.$ must have at least one limiting point $\bar{x}$ belonging to the interval $(\alpha, \beta)$, for at most a finite number of the $x_i$'s can coincide. But $\bar{x}$ would then be a $X$-point, and thus the above assertion is seen to be true.

From among all the quantities that can serve in the capacity of the above $B$'s, let that one be picked out which is the lower limit of the $B$'s and let it be denoted, for it is also a $B$, by $B'$. Then

$$|S_n(x)| \leqq B', \qquad \alpha \leqq x \leqq \beta, \qquad n = 1, 2, \ldots.,$$

while

$$|S_n(x)| > B' - \varepsilon$$

for some values of $n$, $x$, if $\varepsilon > 0$.

Hitherto $n$ has begun with the value 1. If now it begin with the value $m \geqq 1$: $n = m, m+1, \ldots.$, the corresponding $B'$, which shall be denoted by $B'_m$, will never increase with $m$, but it may decrease. Let

$$\lim_{m=\infty} B'_m = \bar{B}.$$

*Coefficient of Convergence.* We have already spoken of the *strength* of the non-uniformity of the convergence in an interval (§2, end). It is desirable to make this conception precise by defining the magnitude $\bar{B}$ as the *Coefficient of the Convergence in the Interval* $(\alpha, \beta)$ and to take this coefficient as measuring the strength of the non-uniformity of the convergence. Its value will then characterize the quality of the convergence in the interval as a whole. Thus when $\bar{B} = 0$ the convergence will be uniform, and conversely.

A simple example to illustrate the foregoing will not be out of place. Let

$$s_n(x) = \frac{n+1}{n} \cdot \frac{nx}{1 + n^2x^2} = S_n(x).$$

$S_n(x)$ has only one peak (Fig. 8) and its altitude is $\frac{n+1}{n}\cdot\frac{1}{2}$. Hence $B'=1$; $B'_m=\left(1+\frac{1}{m}\right)\cdot\frac{1}{2}$; $\bar{B}=\frac{1}{2}$.

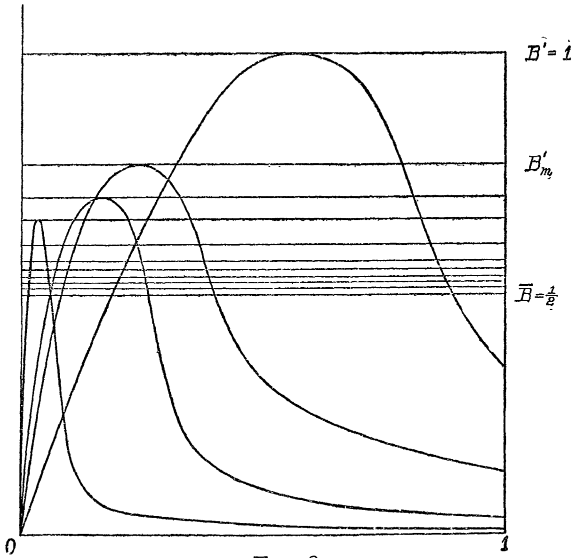


FIG. 8.

6. In the preceding paragraph a necessary condition has been established for the set of points in whose neighborhood $s_n(x)$ has infinite peaks (the $X$-points). In the present paragraph a necessary condition shall be established for the set of points in whose neighborhood $s_n(x)$ has null peaks (the $\zeta$-points).

*Definition of $\zeta$-Points.* If a point $x_0$ is such that, after the arbitrary choice of a positive quantity $\varepsilon$, an interval $(x_0-\delta,\ x_0+\delta)$ can be determined whose coefficient of convergence $\bar{B}$ is less than $\varepsilon$, such a point $x_0$ shall be denoted by $\zeta$ and designated as a $\zeta$-point.

THEOREM II.—*The $\zeta$-points form a set of points that is* AT LEAST *dense throughout $L$.*

If this were not the case there would be an interval $(\alpha, \beta)$ in $L$ in which no $\zeta$-point appears. Let $\eta_1, \eta_2, \ldots.$ be a set of constantly decreasing positive numbers with $\lim\limits_{i=\infty}\eta_i=0$. Begin with $\eta_1$ as the $A$ and $(\alpha, \beta)$ as the interval of the Fundamental Theorem. Then there will be a sub-interval $(\alpha_1, \beta_1)$ of $(\alpha, \beta)$ no one of whose points is a $\gamma$-point and it follows from reasoning similar to that employed in §5 that

$$|S_n(x)|<\eta_1, \qquad \alpha_1\leqq x\leqq\beta_1$$

for all values of $n$ from a fixed integer $m_1$ on. Hence the coefficient of convergence for this interval, $\overline{B}_1$, is not greater than $\eta_1$.

Next repeat the above step, taking as the $A$ of the Fundamental Theorem $\eta_2$ and as the interval $(\alpha_1, \beta_1)$. Then an interval $(\alpha_2, \beta_2)$ lying within $(\alpha_1, \beta_1)$, and an integer $m_2 \geqq m_1$ will be found such that

$$|S_n(x)| < \eta_2, \qquad \alpha_2 \leqq x \leqq \beta_2, \qquad n \geqq m_2.$$

Successive repetitions of this step lead to a series of intervals $(\alpha_i, \beta_i)$, where $\alpha_i < \beta_i$, $\alpha < \alpha_1 < \alpha_2, \ldots,$ $\beta > \beta_1 > \beta_2, \ldots$; and a series of integers $m_1 \leqq m_2 \leqq m_3 \ldots$ such that

$$|S_n(x)| < \eta_i, \qquad \alpha_i \leqq x \leqq \beta_i, \qquad n > m_i.$$

The $\alpha$'s and $\beta$'s may be so chosen that

$$\lim_{i=\infty} \alpha_i = \lim_{i=\infty} \beta_i = \bar{c}.$$

Then the point $\bar{c}$ is a $\zeta$-point and the interval $(\alpha, \beta)$ is not free from $\zeta$-points. From the contradiction herein contained follows the truth of the theorem.

7. Uniform and non-uniform convergence are conceptions that relate to the behavior of the variable function $s_n(x)$ *throughout an interval.* The theorems just established have brought out the importance of the role that certain *points* of the interval play with regard to the behavior of the peaks in their vicinity. Let $x_0$ be any point of $L$, $\overline{B}_h$ the coefficient of convergence of the interval $(x_0, x_0 + h)$. $\overline{B}_h$ never increases when $|h|$ decreases, and it is always positive. Hence it converges toward a limit that is positive or zero. Let

$$\lim_{h=0} \overline{B}_h = \bar{b}^+, \quad \bar{b}^-,$$

according as $h$ is positive or negative.

*Indices of a Point.* $\bar{b}^+$, $\bar{b}^-$ shall be defined as the *forward* resp. *backward index* of the point $x_0$.

If the convergence is uniform throughout an interval, $\bar{b}^+ = \bar{b}^- = 0$ at all points of the interval, and conversely, if $\bar{b}^+ = \bar{b}^- = 0$ for each point of an interval, the convergence is uniform throughout the interval.*

* This theorem is virtually contained in the opening paragraph of du Bois' paper above referred to; of course in different form, since du Bois did not have the indices of a point.

With this definition Theorems I and II may be restated as follows:

THEOREM I.—*The points of $L$ one of whose indices is infinite form* AT MOST *a set satisfying Conditions* $(P)$.

THEOREM II.—*The points of $L$ in which each index is* 0 *form* AT LEAST *a set dense throughout $L$.*

8. We have thus been led to two theorems that state necessary conditions for the set of $X$-points and the set of $\zeta$-points. Are these conditions, conversely, sufficient? i. e.

*a*) Given any set $G$ of points satisfying Conditions $(P)$, does there exist a function $s_n(x)$ satisfying Conditions $(A)$ and having the points $G$ (and no others) as its $X$-points?

*b*) Given any set $H$ of points dense throughout $L$, does there exist a function $s_n(x)$ satisfying Conditions $(A)$ and having the points $H$ (and no others) as its $\zeta$-points?

The first of these questions is answered in what follows in the affirmative, the second in the negative, and the complete condition (necessary and sufficient) for the $\zeta$-points is formulated in terms of a certain class of sets of points.

Before proceeding to the proofs just promised, I will give an example useful both here and in the second part of this paper.

Let a set of points $\Gamma$ be constructed as follows.* The interval $L$ shall be the interval (0, 1) (Fig. 9).

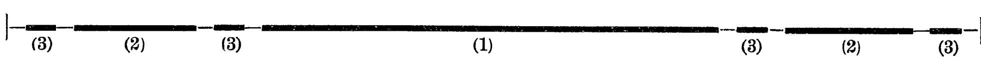


FIG. 9.

*First Step.* In the middle of this interval lay off an interval (1) of length

$$l_1 = \lambda - \tfrac{1}{3}\lambda,$$

where $\lambda$ is chosen arbitrarily as a positive quantity not greater than unity: $0 < \lambda \leqq 1$. For present purposes $\lambda$ shall be taken as $\frac{3}{4}$, $l_1 = \frac{1}{2}$.

*Second Step.* In the middle of each of the free end intervals lay off an inter-

* This is a special case of a set given by Harnack, Math. Ann., vol. 19, p. 239.

val (2), both of these intervals to be of the same length $l_2$ and such that the total length of the intervals (1), (2) is

$$l_1 + 2l_2 = \lambda - \tfrac{1}{4}\lambda.$$

*n-th Step.* In the middle of each of the equal free intervals lay off an interval $(n)$, all of these intervals to be of the same length $l_n$ and such that the total length of the intervals (1), (2), .... $(n)$ is

$$l_1 + 2l_2 + 2^2 l_3 + \dots + 2^{n-1} l_n = \lambda - \frac{\lambda}{n+2}.$$

When $n$ increases indefinitely, a set of intervals is obtained and their extremities form a set of points. If to these points those points of $L$ not already included among these extremities, but which still are limiting points of the extremities, be added and this complete set of points be denoted by $\Gamma$, then $\Gamma$ is an example of the most general set of points satisfying Conditions $(P)$;—the most general, at least, in regard to the properties of such sets that are of importance here.*

A function $s_n(x)$ satisfying Conditions $(A)$ and having the points of $\Gamma$ as its $X$-points shall now be constructed. To begin with, consider the function

$$\psi_n(x) = nxe^{-nx^2}, \qquad x \geqq 0.$$

The approximation curves are of the same character as those shown in Fig. 1. Next form the function†

$$\begin{aligned}\phi_n(x, l) &= \quad \frac{\pi}{l}\sin\frac{\pi x}{l}\,.\,\psi_n\left(\cos\frac{\pi x}{l}\right) \qquad && 0 \leqq x \leqq \frac{l}{2} \\ &= -\frac{\pi}{l}\sin\frac{\pi x}{l}\,.\,\psi_n\left(\cos\frac{\pi x}{l}\right) && -\frac{l}{2} \leqq x \leqq 0 \\ &= 0 \text{ for all other values of } x. &&\end{aligned}$$

---

* The objection might be raised to this example that $\Gamma$ contains no isolated points, each point of $\Gamma$ being a limiting point. It would be easy to meet this objection either by taking a less special case of Harnack's set or by adding to $\Gamma$ the points ($\lambda$ being taken $= \frac{1}{2}$):

$$\tfrac{1}{4} + \frac{1}{9+n},\ \tfrac{1}{2} - \frac{1}{9+n},\ \tfrac{1}{2}; \qquad n = 1, 2, \dots.$$

But the example would thereby be rendered only more complicated without fulfilling any better its mission of illustration for the purposes of this paper.

† The function now to be formed is chosen partly with reference to future use. It would suffice for present purposes to omit the factors $\frac{\pi}{l}\sin\frac{\pi x}{l}$, $-\frac{\pi}{l}\sin\frac{\pi x}{l}$.

The approximation curves for this function are indicated in Fig. 10.

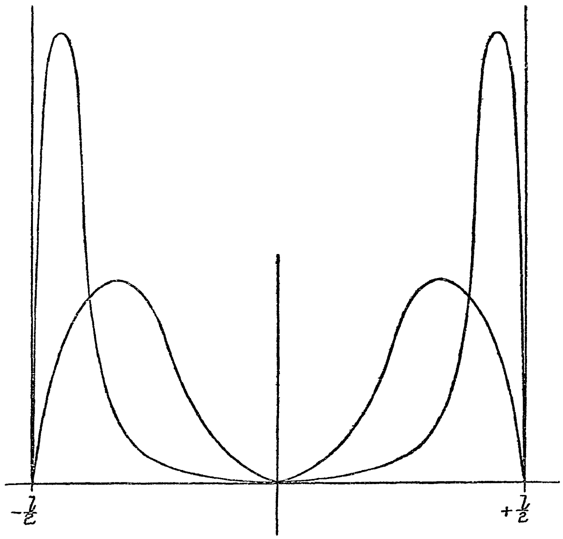

Fig. 10.

Let the middle points of the above intervals $(n)$ be denoted by

$$a_1^{(n)},\ a_2^{(n)},\ \ldots\ldots\ a_{2^n-1}^{(n)}.$$

We are now ready to form our $s_n(x)$, which we define as follows:

$$\begin{aligned} s_n(x) = \quad & \Phi_n(x - a_1^{(1)}, l_1) \\ & + \Phi_n(x - a_1^{(2)}, l_2) + \Phi_n(x - a_2^{(2)}, l_2) \\ & + \Phi_n(x - a_1^{(3)}, l_3) + \ldots\ldots\ldots + \Phi_n(x - a_4^{(3)}, l_3) \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & + \Phi_n(x - a_1^{(n)}, l_n) + \ldots\ldots\ldots\ldots\ldots\ldots + \Phi_n(x - a_{2^n-1}^{(n)}, l_n). \end{aligned}$$

$s_n(x)$ is continuous in $L$ and converges toward 0 for every value of $x$; for if $x_0$ is a point of any interval $(i)$, at most one term in the above sum of terms is different from 0, and this term converges toward 0 for all values of $x$. And if $x_0$ does not lie in any interval $(i)$, all the terms of this sum are 0. Every extremity of an interval is a $X$-point and hence every point of $\Gamma$ and no other point of $L$ is a $X$-point.

*ad a*). By the aid of this example it is now readily seen that if any set of points $G$ satisfying Conditions $(P)$ be given, a function $s_n(x)$ can be constructed satisfying Conditions $(A)$ and having these points as its $X$-points. For let

$\eta_1, \eta_2, \ldots$ be a set of constantly decreasing positive quantities with $\lim_{i=\infty} \eta_i = 0$. Consider the intervals of $L$ which, with the exception of their extremities, are free from points of $G$. There will be but a finite number of such intervals whose length is greater than $\eta_1$. Denote these, taken in any order, as $(1)_1, (1)_2, \ldots (1)_{k_1}$ (Fig. **11**). Next proceed to those not already considered

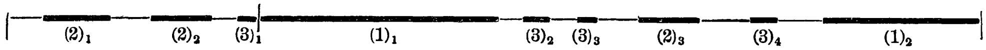

FIG. 11.

whose length is greater than $\eta_2$ and denote them, taken in any order, by $(2)_1, (2)_2, \ldots (2)_{k_2}$. And so on. Let the middle-point and the length of $(n)_m$ be denoted respectively by $a_{n,\,m}$, $l_{n,\,m}$. Then

$$s_n(x) = \sum_{i=1}^{n} \sum_{j=1}^{k_i} \phi_n(x - a_{i,\,j},\ l_{i,\,j})$$

will satisfy Conditions ($A$) and have the points of $G$ as its $X$-points.

9. Before beginning the direct study of question $b$) I will make a digression, the object of which is the ascertainment of certain properties of the set of points of $L$ complementary to the $\zeta$-points.

The $X$-points were obtained (§4) by allowing $A$ to *increase*, thus sifting out from the $\gamma$-points certain ones; and those that always stayed in the sieve were the $X$-points. Here just the reverse process shall demand our attention: $A$ shall *converge toward* 0 and the set of points toward which the corresponding $\gamma$-points converge (in a sense presently to be defined accurately) shall be considered.

Let $\eta_1, \eta_2, \ldots$ be a set of constantly decreasing positive quantities with $\lim_{i=\infty} \eta_i = 0$. Take $\eta_i$ as the $A$ of the Fundamental Theorem and denote the corresponding set of $\gamma$-points by $G_i$. Then all the points of $G_i$ are also points of $G_{i'}$, if $i' > i$; and it is natural to denote that set of points of $L$, each of which appears in some $G_i$ as $\lim_{i=\infty} G_i$. This set I will call $q$ and write

$$q = \lim_{i=\infty} G_i.$$

Thus $q$ may consist of a single point, as in Ex. 1, §2; or it may be dense throughout $L$, as in Ex. 3 of the same paragraph. $q$ contains all $X$-points.

The points of $q$ are those points of $L$ that are not $\zeta$-points; in other words, they are those points whose indices are not both 0.

*Definition of $\xi$-Points.* A point of $q$ shall be denoted by $\xi$ and designated as a $\xi$-point.

$q$ belongs to a class of sets of points of so much importance, as will presently appear (Theorem III), that I will define this class independently of the considerations that have led to it.

*The Class of Sets of Points $Q$.* A set of points such that a variable set of points $G_i$ exists satisfying the following conditions:

1) $G_i$ satisfies Conditions $(P)$;

2) $G_{i'}$ contains *at least* all the points of $G_i$, if $i' > i$;

3) Any point of the given set appears in $G_i$, if $i$ be chosen sufficiently large—

shall be defined as belonging to the Class of Sets of Points $Q$ and denoted as a set of points $Q$.

$$Q = \lim_{i=\infty} G_i.$$

Every set of points satisfying Conditions $(P)$ is obviously a set $Q$.

10. Thus it appears that a necessary condition which the $\xi$-points satisfy is that they form a set of points $Q$. But the converse is also true, and both of these theorems shall be combined in

Theorem III.—*The $\xi$-points form a set of points $Q$; and conversely, to every set of points $Q$ there corresponds a function $s_n(x)$ satisfying Conditions $(A)$ and having $Q$ as its $\xi$-points.*

The proof of the second part of the theorem is as follows. Let $G_\rho$ be any component set of $Q$. Then a function $\sigma_n^{(\rho)}(x)$ satisfying Conditions $(A)$ and having the points of $G_\rho$ as its $\gamma$-points can be constructed in a manner similar to that in which the function $s_n(x)$ of §8 was built up. Out of these elementary functions $\sigma_n^{(\rho)}(x)$ the desired function $s_n(x)$ is forged.

Let
$$\overline{\Psi}_n(x) = \sqrt{2e}\, nxe^{-n^2x^2}, \qquad x \geqq 0.$$

The altitude of the peaks is unity. Let

$$\overline{\Phi}_n(x, l) = \overline{\Psi}_n\left(\cos\frac{\pi x}{l}\right), \qquad -\frac{l}{2} \leqq x \leqq \frac{l}{2};$$
$$= 0 \text{ for all other values of } x.$$

Further, let

$$\sigma_n^{(\rho)}(x) = \sum_{i=1}^{n} \sum_{j=1}^{k_i^{(\rho)}} \overline{\Phi}_n (x - a_{i,j}^{(\rho)}, l_{i,j}^{(\rho)}),$$

where $k_i^{(\rho)}$, $a_{i,j}^{(\rho)}$, $l_{i,j}^{(\rho)}$ stand here in the same relation to $G_\rho$ as $k_i$, $a_{i,j}$, $l_{i,j}$ do in §8 to $G$.

If $s_n(x)$ be now defined by the equation

$$s_n(x) = \sum_{\rho=1}^{n} \frac{1}{\rho\,!} \sigma_n^{(\rho)}(x),$$

then $s_n(x)$ satisfies Conditions ($A$) and has $Q$ for its $\xi$-points and no others.

That the points $Q$ are $\xi$-points is seen by the same reasoning as that employed in Ex. 3, §2. But these are the only $\xi$-points. For let $x_0$ be any other point and choose the positive quantity $\varepsilon$ at pleasure. Then

1) $\bar{\rho}$ can be so taken that

$$\frac{1}{(\bar{\rho}+1)\,!} + \frac{1}{(\bar{\rho}+2)\,!} + \dots < \frac{1}{\bar{\rho}\,!\,.\,\bar{\rho}} < \tfrac{1}{2}\varepsilon,$$

and hence

$$\sum_{\rho=\bar{\rho}+1}^{\infty} \frac{1}{\rho\,!} \sigma_n^{(\rho)}(x) < \tfrac{1}{2}\varepsilon,$$

whatever $x$ may be.

2) $\bar{\rho}$ being thus fixed, a positive quantity $\delta$ can be so determined that all points of $G_{\bar{\rho}}$ (and hence of $G_\rho$, if $\rho < \bar{\rho}$) are external to the interval $(x_0 - \delta, x_0 + \delta)$, and hence $m$ can be so taken that for all points $x$ of this interval

$$\sum_{\rho=1}^{\bar{\rho}} \frac{1}{\rho\,!} \sigma_n^{(\rho)}(x) < \tfrac{1}{2}\varepsilon, \qquad n > m.$$

From the addition of these last two inequalities it follows that

$$s_n(x) < \varepsilon, \qquad x_0 - \delta \leqq x \leqq x_0 + \delta.$$

But $s_n(x)$ is never negative. Thus $x_0$ is a $\zeta$-point and the theorem is proved.

11. *ad b*) Since the $\zeta$-points and the $\xi$-points are complementary to each other, the answer to question *b*) is implied in Theorem III and may be expressed as a corollary to that theorem.

COROLLARY.—*The $\zeta$-points form a set of points whose complementary set is a set $Q$; and conversely, any set of points whose complementary set is a set $Q$ is a possible set of $\zeta$-points.*

Thus the problem of determining whether a given set of points can serve as a set of $\xi$- resp. $\zeta$-points has been shown to be identical with the problem of determining whether the given set resp. its complementary set is a set belonging to the class $Q$; and so the solution of question $b$) has been made to depend on the solution of a problem in the Theory of Sets of Points.

12. Before leaving this subject I will deduce a further necessary condition for the $\zeta$-points.

THEOREM IV.—*The $\zeta$-points are non-enumerable in every subinterval of $L$ whatsoever.*

Since $L$ itself was an arbitrary interval, it is sufficient to show that the $\zeta$-points are not enumerable in $L$. Suppose they were enumerable. Let them be denoted by $\zeta_1, \zeta_2, \ldots$. Add to the points $G_1$ the point $\zeta_1$; to $G_2$, $\zeta_1, \zeta_2$; to $G_i$, $\zeta_1, \zeta_2, \ldots \zeta_i$; and denote the new sets of points respectively by $G_1', G_2', \ldots G_i'$. Then $\lim_{i=\infty} G_i'$ is the *totality* of the points of $L$, and hence, since $G_i'$ satisfies the conditions of a component, this totality of points appears as a set $Q$. But the complementary set is nil, so that that function $s_n(x)$ which by Theorem III has $Q$ as its $\xi$-points, has no $\zeta$-points, and this is absurd.

Thus it appears that not every set of points dense throughout $L$ can serve as a set of $\zeta$-points; e. g. the rational numbers could not form a set of $\zeta$-points. But the $\xi$-points, as is easily shown by an example, may form a set non-enumerable in every interval.

In the proof of Theorem IV we have obtained incidentally a theorem regarding sets $Q$, namely: *A set $Q$ forms in no interval whatsoever a continuum. The complementary set is in every interval non-enumerable.*

Theorems II and IV express two independent necessary conditions for the $\zeta$-points. Whether these conditions are also sufficient is a question demanding further study.

## II.—THE INTEGRATION OF SERIES TERM BY TERM.

13. The question that forms the subject of the second part of this paper is the determination of the conditions under which the $u$-series of §1 can be inte-

grated term by term; i. e. that

$$\int_{x_0}^{x_1} \lim_{n=\infty} s_n'(x)\,dx = \lim_{n=\infty} \int_{x_0}^{x_1} s_n(x)\,dx,$$

when $$a \leqq x_0 < x_1 \leqq b;$$

or expressed in terms of $S_n(x)$, that

$$\int_{x_0}^{x_1} \lim_{n=\infty} S_n(x)\,dx = \lim_{n=\infty} \int_{x_0}^{x_1} S_n(x)\,dx.$$

Geometrically the left-hand side of these equations signifies the area under the curve $y = f(x)$ resp. $y = 0$ between $x = x_0$ and $x = x_1$; the right-hand side, on the other hand, is the *limit* that the corresponding area under the *variable* curve $y = s_n(x)$ resp. $y = S_n(x)$ approaches when $n = \infty$.

Since $\lim\limits_{n=\infty} S_n(x) = 0$, $\int_{x_0}^{x_1} \lim\limits_{n=\infty} S_n(x)\,dx = 0$ and the question of Part II in its reduced form is: *Under what conditions will*

$$\lim_{n=\infty} \int_{x_0}^{x_1} S_n(x)\,dx = 0\,?$$

14. Let us begin with some simple examples.

Ex. 1. $$S_n(x) = nxe^{-nx^2},$$

$$\int_0^x S_n(x)\,dx = \tfrac{1}{2}(1 - e^{-nx^2}),$$

$$\lim_{n=\infty} \int_0^x S_n(x)\,dx = \tfrac{1}{2} \neq 0.$$

Thus although the integrand converges toward 0 for all values of $x$, the integral does not, if $x = 0$ forms one of the limits of the integration. Geometrically the area under the curve $y = nxe^{-nx^2}$ in the interval $(0, x)$ converges toward $\frac{1}{2}$.

If $$S_n(x) = n^2xe^{-nx^2},$$

$$\lim_{n=\infty} \int_0^x S_n(x)\,dx = \infty.$$

If however $$S_n(x) = \sqrt{n}\,xe^{-nx^2},$$

$$\lim_{n=\infty} \int_0^x S_n(x)\,dx = 0,$$

and the same is true if $S_n(x) = \dfrac{n^2x}{1 + n^3x}$.

In the first, second and fourth cases the peaks are infinite (Fig. 1), in the third they are not (Fig. 2).

Ex. 2. $$S_n(x) = \phi_n(x, l), \qquad (\S 8)$$

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} S_n(x)\,dx = 2\int_0^{\frac{l}{2}} \frac{\pi}{l}\, n \sin\frac{\pi x}{l} \cos\frac{\pi x}{l}\, e^{-n\cos^2\frac{\pi x}{l}}\,dx = \int_{-n}^{0} e^{\xi} d\xi = 1 - e^{-n},$$

where
$$\xi = -n\cos^2\frac{\pi x}{l};$$

hence
$$\lim_{n=\infty}\int_{-\frac{l}{2}}^{\frac{l}{2}} S_n(x)\,dx = 1.$$

It thus appears that when there is a single point in $L$ in whose neighborhood peaks occur,

$$\int_{x_0}^{x_1} S_n(x)\,dx$$

may, if the peaks become infinite, converge toward $0$, toward a limit different from $0$, or diverge. If the peaks remain finite, then it is at once evident geometrically that the limit must be $0$.

15. Turning now to the general case, we recall that the $X$-points of $L$ form at most a set nowhere dense in $L$ (Theorem I); and secondly, if an interval $(\alpha, \beta)$ is free from such points, the approximation curves $y = S_n(x)$ will all lie within a finite belt bounded by $y = \pm B$ (§5). If the limits of integration lie within such an interval $(\alpha, \beta)$, then we might be inclined, generalizing from the corresponding example above, to regard it as extremely plausible that in this case $\lim\limits_{n=\infty}\int_{x_0}^{x_1} S_n(x)\,dx = 0$. But a more careful study, in the light of Part I of this paper, of the possible manner of the convergence of $S_n(x)$ toward its limit,* diminishes materially the plausibility of this inference, and in fact the proof here given *that in this case* $\lim\limits_{n=\infty}\int_{x_0}^{x_1} S_n(x)\,dx$ *actually is* $0$, although in its beginnings

* It is of great importance at this point to picture to oneself the successive approximation curves $y = S_n(x)$ of §8. An appreciation of the possibilities thereby brought to light would have saved du Bois from the errors he made in the paper above referred to (§4, foot-note).

suggested by intuition, in its further course makes clear that we have here to do with relations that transcend the bounds of our present intuition.

16. The plan of the proof is as follows. Let the positive quantity $\varepsilon$ be chosen at pleasure; then there exists a corresponding fixed integer $m$ such that

$$-\varepsilon < \int_{x_0}^{x_1} S_n(x)\,dx < \varepsilon, \qquad n > m.$$

For if this were not the case, one of the inequalities, say the right-hand one, would be violated for an infinite sequence of values of $n$: $n_1, n_2, \ldots$. Choose the interval of the Fundamental Theorem as $(x_0, x_1)$ and take $A < \frac{\varepsilon}{x_1 - x_0}$. Then those portions of the approximation curve

$$y = S_{n_i}(x), \qquad i = 1, 2, \ldots.$$

which lie above the line $y = A$ would, together with the corresponding segments of this line bound an area (the shaded area of Fig. 12) which could not converge toward 0 when $i$ increases. For

$$\int_{x_0}^{x_1} S_{n_i}(x)\,dx \geqq \varepsilon > A.(x_1 - x_0)$$

and $A.(x_1 - x_0)$ is the area below the line $y = A$. I show however that this latter inequality is impossible (i. e. that the shaded area of Fig. 12 *must* converge

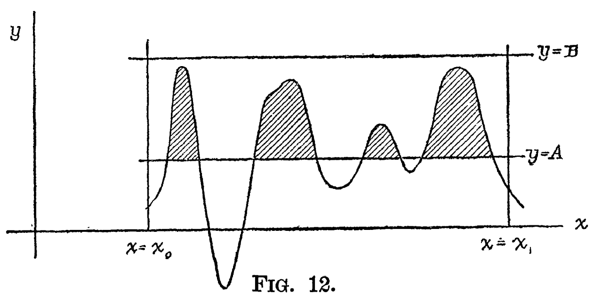

FIG. 12.

toward 0). Hence the above supposition is untenable and from this the truth of the right-hand inequality follows. The same reasoning applies to the left-hand inequality. But this double-inequality is only another form of statement of the theorem itself.

17. A preliminary study of some further properties of the most general set of points $G$ satisfying Conditions $(P)$ is necessary.

We have already enumerated the intervals $(n)_m$ of $L$ (§8) whose extremities are points of $G$, but which contain in their interior no further points of $G$. Let

$$l_{n,1} + l_{n,2} + \dots + l_{n,k_n} = \lambda_n,$$

$$\sum_{n=1}^{\infty} \lambda_n = \lambda.$$

$\lambda$ is surely not greater than $l$, but, as in the example of §8, it may be less. The points of $G$ may be enclosed in a finite number of intervals, the sum of whose lengths exceeds $l - \lambda$ by less than the arbitrary quantity $\delta$. For let $p$ be so taken that

$$\sum_{i=p+1}^{\infty} \lambda_i < \tfrac{1}{2}\delta.$$

Then

$$\sum_{i=1}^{p} \lambda_i > \lambda - \tfrac{1}{2}\delta.$$

Out of each of the $\mu$ intervals whose sum is $\sum_{i=1}^{p} \lambda_i$ a subinterval $\theta_\rho$ may be selected, neither of whose extremities coincides with those of the interval in which it lies and of such length that

$$\sum_{\rho=1}^{\mu} \theta_\rho > \sum_{i=1}^{p} \lambda_i - \tfrac{1}{2}\delta > \lambda - \delta.$$

The interval $L$ is thus divided into $\mu$ subintervals $\theta_\rho$, the sum of whose lengths

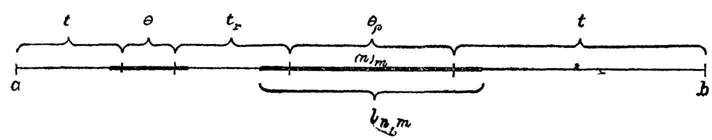


FIG. 13.

lies between $\lambda$ and $\lambda - \delta$ and which, inclusive of their extremities, contain no point of $G$. Thus the points of $G$ fall in the remaining intervals, $t_r$, let us call them, finite in number, whose sum $\sum t_r$ lies between $l - \lambda$ and $l - \lambda + \delta$. And if any set of intervals, finite in number, be so taken as to enclose in their interiors (not on their boundaries) all the points of $G$, then their sum must be greater

than $l - \lambda$; for the above set of $\theta$-intervals can always be so taken that all points of the corresponding $t$-intervals lie within this set.

On the ground of these relations, $l - \lambda$ is defined as the *content** of $G$. Let it be denoted by $I$:

$$I = l - \lambda.$$

18. LEMMA. Let $G$ be any set of points satisfying Conditions $(P)$, $G_i$ a component of $G$:

$$\lim_{i=\infty} G_i = G;$$

and let the content of $G$, $G_i$ be denoted respectively by $I$, $I_i$. Then

$$\lim_{i=\infty} I_i = I.$$

This theorem is far from being self-evident. Suppose we had, instead of $G$, the set of rational numbers $R$ in the interval $(0, 1)$ and developed this set into the series

$$R = R_1 + R_2 + \ldots.,$$

where $R_1 = (\frac{1}{2})$, $R_2 = (\frac{1}{3}, \frac{2}{3})$, $R_3 = (\frac{1}{4}, \frac{3}{4})$; $R_4 = (\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5})$, etc.,

$$G_i = R_1 + R_2 + \ldots. + R_i.$$

Then $$I_i = 0; \text{ but } I = 1.$$

The lemma is proved as follows: Since $I_i \leqq I$ and $I_{i'} \geqq I_i$, if $i' > i$, $I_i$ approaches a limit when $i$ becomes infinite. Let $\lim\limits_{i=\infty} I_i = I'$. Then $I' \leqq I$.

Let the positive quantity $\bar{\delta}$ be chosen at pleasure and then let the constantly decreasing quantities $\bar{\delta}_1, \bar{\delta}_2, \ldots.$ be so taken that

$$\bar{\delta}_1 + \bar{\delta}_2 + \ldots. = \bar{\delta}.$$

The points of $G_1$ can be enclosed in a finite number of intervals $\tau_\kappa^{(1)}$ whose sum $\sum\limits_\kappa \tau_\kappa^{(1)}$ is less than $I_1 + \bar{\delta}_1$ and whose extremities are not points of $G$; the neighborhood of each extremity will then also be free from the points of $G$. The points of $G_2$ not already lying in the intervals $\tau_\kappa^{(1)}$ can be enclosed in a finite

* Harnack, Math. Ann., vol. 25; Cantor, ibid. vol. 23, p. 473.

number of further intervals $\tau_\kappa^{(2)}$ whose extremities are not points of $G$ and such that

$$\sum_\kappa \tau_\kappa^{(1)} + \sum_\kappa \tau_\kappa^{(2)} < I_2 + \overline{\delta}_1 + \overline{\delta}_2.$$

For it becomes evident on a little reflection that if $I_2'$, $I_2''$ denote respectively the content of that portion of $G_2$ lying in the intervals $\tau_\kappa^{(1)}$, $\tau_\kappa^{(2)}$, then $I_2' + I_2'' = I_2$. And $I_2' \geqq I_1$. It is sufficient therefore to make $\sum_\kappa \tau_\kappa^{(2)} < I_2'' + \overline{\delta}_2$.

The repetition of this step leads to a set of intervals $\tau$, finite in number, which include in their interior all the points of $G_i$ and whose extremities, not being themselves points of $G$, lie each within an interval free from points of $G$; and furthermore, the intervals $\tau_\kappa^{(j)}$ remain unchanged for all values of $i > j$, new intervals $\tau_\kappa^{(i')}$, $i' > j$, being merely added. Finally

$$\sum_\kappa \tau_\kappa^{(1)} + \sum_\kappa \tau_\kappa^{(2)} + \ldots + \sum_\kappa \tau_\kappa^{(i)} < I_i + \overline{\delta}_1 + \overline{\delta}_2 + \ldots + \overline{\delta}_i < I' + \overline{\delta}.$$

As $i$ increases indefinitely, the number of the $\tau$-intervals does not become infinite, but reaches a certain maximum number $M$. For if the number were to become infinite, the extremities of these intervals would have to cluster about at least one limiting point $x'$, and it would be possible to choose out of the intervals that thus cluster about $x'$ a set of points of $G$ with $x'$ as their limiting point. Hence $x'$ would itself be a point of $G$. But $x'$ lies in no $\tau$-interval, and thus a contradiction results from the supposition that the number of the $\tau$-intervals is infinite.

The $M$ $\tau$-intervals contain in their interior (i. e. exclusive of their extremities) all the points of $G$. For if, as $i$ increases, the points of $G_i$ in a certain interval were to converge toward the extremity of the interval, then this point would also be a point of $G$. Thus all the points of $G$ are enclosed within $m$ intervals, the sum of whose lengths is less than $I' + \overline{\delta}$, a quantity which, if $I' < I$, can by proper choice of $\overline{\delta}$ be made less than $I$. But this is impossible (§17). Hence $I' = I$.

The set of points $G$ to which the lemma is to be applied is the set of $\gamma$-points of §3, the component $G_i$ being obtained as follows. Let $\gamma$ denote an arbitrary point of $G$ and let $j$ be the smallest integer for which

$$|S_n(\gamma)| \leqq A, \qquad n \geqq j.$$

Then those $\gamma$-points corresponding to values of $j \leqq i$ form a set $G_i$ satisfying Conditions $(P)$. For, being points of $G$, they are nowhere dense and evidently any point $\overline{\gamma}$ toward which such a set of $\gamma_i$-points converges is itself a $\gamma_i$-point.

19. We are now ready to enter on the immediate proof that the inequality

$$\int_{x_0}^{x_1} S_{n_i}(x)\, dx \geqq \varepsilon > A.(x_1 - x_0), \qquad i = 1, 2, \ldots .$$

is impossible. Let $\varepsilon_1$ be so taken that

$$\varepsilon_1 < \varepsilon - A.(x_1 - x_0).$$

It is sufficient to show that the shaded area lying above the line $y = A$ of Fig. 12, which shall be denoted by $C_n$, becomes and remains less than $\varepsilon_1$ as $n$ increases, since whatever area lies below this line is surely less than $A(x_1 - x_0)$.

First choose $\delta$ at pleasure, construct the corresponding $\theta$- and $t$-intervals, and then take $m$ so that

$$S_n(x) \leqq A, \qquad n \geqq m$$

for all points $x$ lying in the $\theta_\rho$-intervals. In the most unfavorable class of cases (illustrated in Fig. $14a$. In Figs. $14a$, $14b$ the $\gamma$-points are plotted on the line

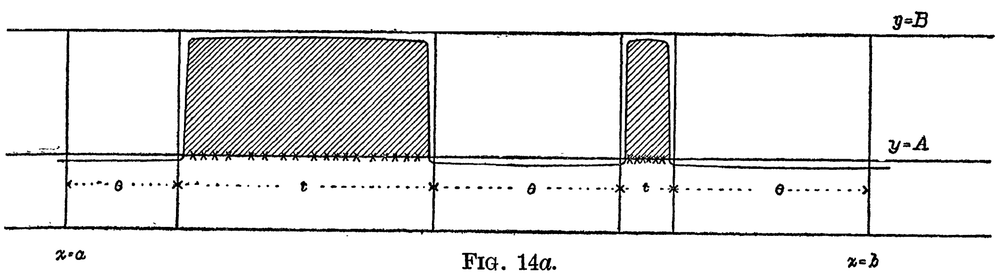


FIG. $14a$.

$y = A$ instead of on the $x$-axis in order to indicate more clearly for what ones of these points $S_n(x)$ is greater and for what ones less than $A$.)

$$G_1 = G_2 = \ldots . = G_m = 0, \qquad I_m = 0,$$

the curve $y = S_m(x)$ spanning all the points of $G$ and rising so abruptly as to make $C_m$ come indefinitely near to the value

$$(B - A).\sum_r t_r < (B - A)(I + \delta).$$

Next let $n = m + p$. The curve $y = S_{m+p}(x)$ (Fig. 14*b*) lies above the

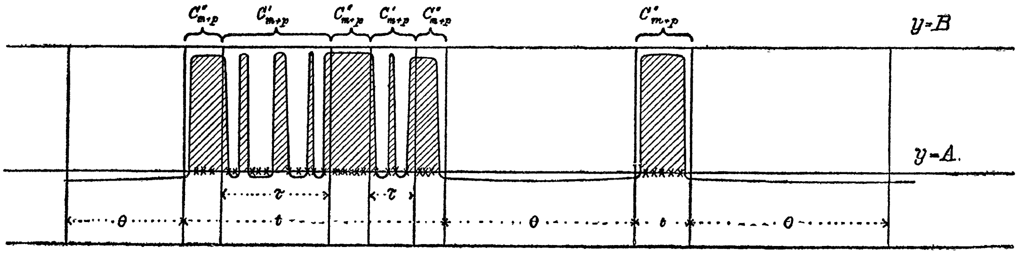


FIG. 14*b*.

line $y = A$ only in the $t$-intervals. Let the $\tau$-intervals be constructed as in §18, $\overline{\delta}$ being assumed arbitrarily, but let them be so taken as always to lie within the $t$-intervals already assumed.

$$I_{m+p} < \sum_\kappa \tau_\kappa^{(1)} + \sum_\kappa \tau_\kappa^{(2)} + \ldots\ldots + \sum_\kappa \tau_\kappa^{(m+p)} < I_{m+p} + \overline{\delta}.$$

Erect perpendiculars to the $x$-axis at the extremities of the $\tau$-intervals. $C_{m+p}$ is thereby divided into two parts: $C'_{m+p}$ lying above the $\tau$-intervals and $C''_{m+p}$ lying above the complementary intervals.

First consider any $\tau$-interval. The curve $y = S_{m+p}(x)$ can rise above the line $y = A$ only in such portions of the interval as are free from points of $G_{m+p}$. If such intervals be added together for all the intervals $\tau$, then the *limit* of their sum is but

$$\sum_\kappa \tau_\kappa^{(1)} + \sum_\kappa \tau_\kappa^{(2)} + \ldots\ldots + \sum_\kappa \tau_\kappa^{(m+p)} - I_{m+p} < \overline{\delta},$$

this inequality holding for all values of $p$. The area under a continuous curve is the limit approached by the sum of the inscribed rectangles. The portions of the rectangles inscribed in $y = S_{m+p}(x)$ that lie above the line $y = A$ have an area less than

$$(B - A)\Big(\sum_\kappa \tau_\kappa^{(1)} + \ldots\ldots + \sum_\kappa \tau_\kappa^{(m+p)} - I_{m+p}\Big).$$

Hence their limit $C'_{m+p}$ cannot exceed this quantity, and so

$$C'_{m+p} < (B - A)\overline{\delta}.$$

On the other hand, the sum of the lengths of the complementary intervals is

$$\sum_r t_r - \Big(\sum_\kappa \tau_\kappa^{(1)} + \ldots\ldots + \sum_\kappa \tau_\kappa^{(m+p)}\Big) < I + \delta - I_{m+p},$$

and hence $$C''_{m+p} < (B-A).(I+\delta-I_{m+p}).$$

Therefore since $C'_{m+p}+C''_{m+p}=C_{m+p}$,

$$C_{m+p} < (B-A).(I-I_{m+p}+\delta+\bar{\delta}).$$

Let the positive quantity $\eta$ be taken at pleasure. Then by the lemma $p_1$ can be so taken that

$$I-I_{m+p} < \eta, \qquad p > p_1.$$

Hence if

$$\eta+\delta+\bar{\delta} < \frac{\varepsilon_1}{B-A},$$

$$C_n < \varepsilon_1, \qquad n > m+p_1.$$

In a similar manner it is shown that the first inequality of §16 is impossible.

Thus the theorem of §15 has been established. It may be stated as follows:

THEOREM V.—*If $s_n(x)$ is any function of $x$ satisfying Conditions (A), $(\alpha, \beta)$ any interval free from X-points, and $x_0 < x_1$ any two points of this interval, then*

$$\int_{x_0}^{x_1} \lim_{n=\infty} s_n(x).dx = \lim_{n=\infty} \int_{x_0}^{x_1} s_n(x)\,dx.$$

*If $s_n(x)$ is given in the form of a series*

$$s_n(x) = u_1(x)+u_2(x)+\ldots\ldots+u_n(x),$$

*the series is said to be integrable term by term.*

But more than this: *It follows from the method above set forth for the proof of this theorem that, if $x_0$, $x_1$ are regarded as variable,*

$$\int_{x_0}^{x_1} s_n(x)\,dx$$

*converges* UNIFORMLY *toward its limit when $n$ becomes infinite.*

For let $S_n(x) = S_n^+(x) + S_n^-(x)$, where $S_n^+(x) = S_n(x)$ when $S_n(x) \geqq 0$ and vanishes for all other values of $x$. $S_n^-(x) \leqq 0$. If $m_1$ be so taken that

$$\int_{\alpha}^{\beta} S_n^-(x)\,dx < \varepsilon, \qquad n > m_1,$$

then

$$\int_{x_0}^{x_1} S_n(x)\,dx < \varepsilon, \qquad \alpha \leqq x_0 < x_1 \leqq \beta.$$

If $m_2$ be so taken that

$$-\varepsilon < \int_{\alpha}^{\beta} S_n^-(x)\,dx, \qquad n > m_2,$$

then

$$-\varepsilon < \int_{x_0}^{x_1} S_n(x)\,dx.$$

And if the larger of the integers $m_1$, $m_2$ be denoted by $m$, then

$$-\varepsilon < \int_{x_0}^{x_1} S_n(x)\,dx < \varepsilon, \qquad \alpha \leqq x_0 < x_1 \leqq \beta, \qquad n > m.$$

20. The question of §13 having been answered for the case that the interval $(\alpha, \beta)$ is free from $X$-points, we now proceed to the case that $X$-points are present. Here the integral may diverge as $n$ increases. If it converges, let

$$F(x, x_0) = \lim_{n=\infty} \int_{x_0}^{x} S_n(x)\,dx.$$

If $x_0$, $x$ lie in an interval $(\alpha, \beta)$ that contains only a finite number of $X$-points, and if $F(x, x_0)$ does not vanish for all values of $\alpha \leqq x \leqq \beta$, then evidently (as in the Exs. of §14) $F(x, x_0)$ will in general be continuous, but may have a finite discontinuity, as indicated in Fig. 15, at one or more of the $X$-points. But the

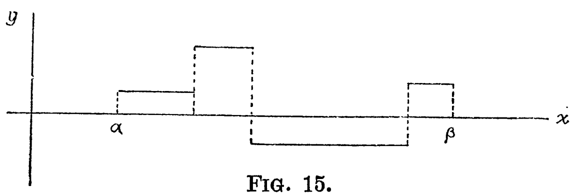


FIG. 15.

consideration of the most general set $G$ of $X$-points (§8) suggests the question: *Is it possible for $F(x, x_0)$ to be a continuous function of $x$ throughout $(\alpha, \beta)$ without vanishing for all values of $x$?*

I answer this question in the *affirmative* by giving an example of such a function. Let the set of points $\Gamma$ and the intervals $(i)$ be the same as in the example of §8, but let

$$s_n(x) = \nu\phi_n(x - a_1^{(n)}, l_n) + \ldots\ldots + \nu\phi_n(x - a_{2^n-1}^{(n)}, l_n),$$

where $\nu = \frac{1}{2^n - 1}$. This amounts to throwing the peaks all into the intervals $(n)$, all other intervals being free.

It is necessary to show 1) that $F(x, x_0)$ exists, i. e. that $\int_{x_0}^{x} s_n(x)\,dx$ converges toward a limit for all values of $x$, $x_0$ belonging to $L$; 2) that $F(x, x_0)$ is a continuous function of $x$ throughout the interval.

Since

$$\int_0^x - \int_0^{x_0} = \int_{x_0}^{x},$$

it is sufficient to show that $\int_0^x s_n(x)\,dx$ converges toward a limit $F(x)$ which is a continuous function of $x$.

The symmetry with which the function $s_n(x)$ was constructed will now stand us in good stead. In fact we see at once from the expressions already obtained for the area under each of the component curves $\nu\phi_n(x - a_i^{(n)}, l_n)$ that go to make up $s_n(x)$, namely $\nu$, that

$$\int_0^1 s_n(x)\,dx = 1\,; \qquad \therefore\ F(1) = 1. \qquad \text{And } F(0) = 0.$$

Furthermore, from the symmetry of $y = s_n(x)$ with regard to the line $x = \frac{1}{2}$, it is evident that

$$\int_0^{\frac{1}{2}} s_n(x)\,dx = \tfrac{1}{2}\int_0^1 s_n(x)\,dx = \tfrac{1}{2}\,; \qquad \therefore\ F(\tfrac{1}{2}) = \tfrac{1}{2}.$$

Let $x$ be any point of the interval (1).

$$\int_0^x s_n(x)\,dx = \int_0^{\frac{1}{2}} s_n(x)\,dx + \int_{\frac{1}{2}}^x s_n(x)\,dx.$$

The second of these integrals is 0 for all values of $n$ greater than 1, and hence

$$\int_0^x s_n(x)\,dx = \tfrac{1}{2}, \qquad n > 1\,;$$

$$\therefore\ F(x) = \tfrac{1}{2}, \qquad a_1^{(1)} - \frac{l_1}{2} \leqq x \leqq a_1^{(1)} + \frac{l_1}{2}.$$

It is possible to generalize this result at once for any interval. Let it be the $j^{\text{th}}$ of the intervals $(i)$, and let $x$ be any point of this interval. Then

$$\int_0^x s_n(x)\,dx = \int_0^{a_j^{(i)}} s_n(x)\,dx + \int_{a_j^{(i)}}^x s_n(x)\,dx.$$

Now, as soon as $n$ is greater than $i$, $s_n(x) = 0$ throughout each interval $(i)$, and the second integral on the right vanishes.

Consider the first integral

$$\int_0^{a_j^{(i)}} s_n(x)\,dx.$$

The intervals (1), (2), (3), .... $(i)$ divide the interval $L$ up into $K_i$ complementary intervals. Let $r_{i,j}$ denote the number of these intervals to the left of $a_j^{(i)}$. As soon as $n$ exceeds the value $i$, the total area (unity) under $s_n(x)$ in the whole

interval $L$ is apportioned equally to each of these complementary intervals. Hence

$$\int_0^{a_j^{(i)}} s_n(x)\,dx = \frac{r_{i,j}}{K_i}, \qquad n > i,$$

and therefore

$$\lim_{n=\infty}\int_0^{a^{(i)}} s_n(x)\,dx = \frac{r_{i,j}}{K_i}.$$

Thus

$$F(x) = \frac{r_{i,j}}{K_i}, \qquad a_j^{(i)} - \frac{l_i}{2} \leqq x \leqq a_j^{(i)} + \frac{l_i}{2}$$

and $F(x)$ exists and is continuous for all values of $x$ within any interval $(i)$, and at the extremity it is at least continuous on the side on which the interval lies. Moreover it is evident from the foregoing that if $x_0$, $x_1$ are any two values of $x$ for which $F(x)$ has already been shown to exist and if

$$x_0 < x_1, \text{ then } F(x_0) \leqq F(x_1).$$

Finally let $0 < x' < 1$ be any point not belonging to any interval $(i)$. Then in general $x'$ lies between two of the intervals $(i)$. Let the middle points of these intervals be denoted by $\alpha_i$, $\beta_i$, where $\alpha_i < \beta_i$. If however no one of the intervals $(i)$ lies to the left (right) of $x'$, let $\alpha_i = 0$ ($\beta_i = 1$). Then

$$\int_0^{\alpha_i} s_n(x)\,dx \leqq \int_0^{x'} s_n(x)\,dx \leqq \int_0^{\beta_i} s_n(x)\,dx,$$

and this double inequality can be written, as soon as $n > i$:

$$F(\alpha_i) < \int_0^{x'} s_n(x)\,dx < F(\beta_i).$$

As $i$ increases $\alpha_i < x'$ and $\beta_i > x'$ both converge toward $x'$ as their limit; for $x'$ being a point of no interval $(i)$, the intervals $(i)$ cluster about $x'$ on both sides. Let $\alpha_i'$ denote the largest of the quantities $\alpha_1, \alpha_2, \ldots . \alpha_i$; $\beta_i'$, the smallest of the quantities $\beta_1, \beta_2, \ldots . \beta_i$. Then

$$\alpha_1' \leqq \alpha_2' \leqq \ldots . < x', \ \lim_{i=\infty} \alpha_i' = x'; \qquad \beta_1' \geqq \beta_2' \geqq \ldots . > x', \ \lim_{i=\infty} \beta_i' = x',$$

$$F(\alpha_i') < \int_0^{x'} s_n(x)\,dx < F(\beta_i'), \qquad n > i.$$

And

$$F(\alpha_1') \leqq F(\alpha_2') \leqq \ldots . ; \quad F(\beta_1') \geqq F(\beta_2') \geqq \ldots .$$

Thus it appears that $F(\alpha_i')$, $F(\beta_i')$ both converge towards limits as $i$ becomes infinite; and since

$$F(\beta_i') - F(\alpha_i') \leqq F(\beta_i) - F(\alpha_i) = \frac{1}{K_i},$$

these limits are equal. Therefore $\int_0^{x'} s_n(x)\,dx$ converges toward the same limit and $F(x')$ exists.

Thus $F(x)$ exists for all values of $x$. It is easily seen that, generally, if

$$0 \leqq x_0 < x_1 \leqq 1, \quad F(x_0) \leqq F(x_1).$$

At the above point $x'$, $F(x)$ is continuous, for

$$\lim_{i=\infty} F(\alpha_i') = F(x'),$$

and if

$$\alpha_i' < x < x',$$

$$F(\alpha_i') \leqq F(x) \leqq F(x').$$

Hence

$$\lim_{x=x'} F(x) = F(x'),$$

$x$ being always less than $x'$. It is shown in a similar manner that if $x > x'$, the same equation holds.

It has been shown that if $x''$ is the extremity of one of the intervals $(i)$, $F(x)$ is continuous at $x''$ on the side on which the interval lies. It remains to show that $F(x)$ is continuous on the other side also, and that $F(x)$ is continuous for $x = 0$ and $x = 1$. The method above employed for the case of the point $x'$ is immediately applicable, the only modification being that only one half of the double inequality is necessary.

The function $F(x)$ is therefore a continuous function of $x$ which never decreases as $x$ increases and such that $F(0) = 0$, $F(1) = 1$. Thus the answer promised to the question raised at the beginning of this paragraph has been given.

It may be remarked that $F(x)$ is constant throughout each of the $\theta$-intervals and hence those points $x'$ in whose neighborhood $F(x) \neq F(x')$ can be enclosed in a finite number of $t$-intervals whose sum is less than $l - \lambda + \delta$, $\delta$ being an arbitrarily chosen positive quantity. If $\lambda$ is taken equal to $l$, this sum will be less than $\delta$, i. e. arbitrarily small.

21. We have already seen that when the $X$-points are finite in number, $F(x, x_0)$ cannot be continuous unless it vanishes for all values of $x$. Evidently the same would be true if the $X$-points were the points $\frac{1}{n}$, $0$. For, $F(x, x_0)$ being assumed to exist for all values of $x$, $x_0$ in the interval $(0, 1)$, it is easily shown that

$$F(x, 0) = F(x_0, 0) + F(x, x_0).$$

If $x_0 > 0$, $F(x, x_0) = 0$, there being but a limited number of $X$-points in the interval $(x_0, x)$. Further,

$$F(x, 0) = \lim_{x_0 = 0} F(x_0, 0) = 0$$

and hence if $F(x, x_0)$ is continuous, it vanishes for all values of $x$.

From this example we can clearly generalize as follows. $F(x, x_0)$ cannot be continuous unless it vanishes for all values of $x$, if the derivative $G^{(1)}$ of the set $G$ of $X$-points consists of a finite number of points, i. e. if the second derivative $G^{(2)} \equiv 0$.

But even if $G^{(1)}$ contains an infinite number of points, the same will be true, provided $G^{(2)}$ contains but a finite number of points, i. e. that $G^{(3)} \equiv 0$. And by induction we infer that $F(x, x_0)$ cannot be continuous unless it vanishes for all values of $x$, if the set of $X$-points is *reducible*; i. e. if $G^{(\alpha)} \equiv 0$, where $\alpha$ denotes a number of the first or second class.*

Suppose on the other hand the set $G$ of $X$-points is *perfect*.* Then I say it is always possible for $F(x, x_0)$ to be continuous without vanishing for any other value of $x$ than the value $x = x_0$. Let $G$ be included in the interval $(a, b)$, $L$, where $a, b$ are points of $G$. Select at pleasure an interval (1) whose extremities are points of $G$ not coinciding with $a, b$, but which contains no other points of $G$. Out of each of the free intervals complementary to (1) select, in the same manner as (1) was selected from $L$, intervals $(2)_1$, $(2)_2$. Repeat this process, obtaining the set $(i)_1, (i)_2, \ldots . (i)_{2^i-1}$. Then the function

$$s_n(x) = \sum_{j=1}^{2^n-1} \nu\phi_n(x - a_j^{(n)}, l_{n,j})$$

is such that $F(x, x_0)$ is continuous without vanishing for any other value of $x$ than $x_0$. The proof is similar to that given in §20. The $X$-points of $s_n(x)$ are all points of $G$, and if the intervals $(i)_k$ are so taken that each subinterval of $L$ that contains in its interior no points of $G$ appears in this list, each point of $G$ will be a $X$-point.

Now Cantor has shown* that if the first derivative $P^{(1)}$ of a set of points $P$ is enumerable, there always exists a number $\alpha$ of the first or second class such that $P^{(\alpha)} \equiv 0$; but if $P^{(1)}$ is non-enumerable, then $P^{(1)}$ contains among its points

*Cantor, Fondaments d'une théorie générales des ensembles, Acta Math., vol. 2, p. 405; Sur divers théorèmes de la théorie des ensembles, etc., ibid. p. 409 et seq.; Bendixson, Quelques théorèmes de la théorie des ensembles de points, ibid. p. 419.

a perfect set of points: $P^{(1)} = R + S$, where $R$ does not concern us and $S$ is perfect. Since $G$ contains its derivative $G^{(1)}$, $G$ will be enumerable when* and only when $G^{(1)}$ is enumerable. We are thus led to the following theorem.

THEOREM VI.—*If* $s_n(x)$ *is a function satisfying Conditions* $(A)$ *and* $\lim\limits_{n=\infty}\int_{x_0}^{x} s_n(x)\,dx = \overline{F}(x, x_0)$; *then it is a sufficient condition for the existence of the relation:*

$$\int_{x_0}^{x}\left[\lim_{n=\infty} s_n(x)\right]dx = \lim_{n=\infty}\left[\int_{x_0}^{x} s_n(x)\,dx\right]$$

*that* 1) $\overline{F}(x, x_0)$ *be a continuous function of* $x$ *and* 2) *the set* $G$ *of* $X$*-points of the interval* $(a, b)$ *be enumerable.*

*But if the set* $G$ *of* $X$*-points is non-enumerable, then there always exist functions* $s_n(x)$ *satisfying Conditions* $(A)$ *and having* $G$ *as their* $X$*-points, for which* $\overline{F}(x, x_0)$ *is continuous, but*

$$\int_{x_0}^{x}\left[\lim_{n=\infty} s_n(x)\right]dx \neq \lim_{n=\infty}\left[\int_{x_0}^{x} s_n(x)\,dx\right]$$

*if* $x \neq x_0$.

*If* $s_n(x)$ *be given as a series:*

$$s_n(x) = u_1(x) + u_2(x) + \dots + u_n(x),$$

*then it is a sufficient condition for the integrability of the series term by term that* 1) *the series*

$$\int_{x_0}^{x} u_1(x)\,dx + \int_{x_0}^{x} u_2(x)\,dx + \dots$$

*be a continuous function of* $x$; 2) *the* $X$*-points be enumerable.*

22. The theorems obtained in Part II of this paper lead to corresponding theorems regarding the differentiation of a series term by term. For example:

THEOREM.—*If* $U_i(x)$ *has for all values of* $x$ *in the interval* $(a, b)$: $a \leqq x \leqq b$ *a derivative* $U_i'(x) = u_i(x)$ *continuous at each point of the interval and if the series*

$$U(x) = U_1(x) + U_2(x) + \dots$$

*converges for all values of* $x$ *in the interval toward the limit* $U(x)$; *if further the series of the derivatives*

$$u(x) = u_1(x) + u_2(x) + \dots$$

*Cantor, Sur les ensembles infinis et linéaires de points, ibid. p. 374.

*converges for all values of $x$ toward the continuous function $u(x)$; then in general the function $U(x)$ will have a derivative given by the differentiation of the U-series term by term. The points for which this fails to be true form a set $G$ satisfying Conditions $(P)$.*

*And conversely, if $U(x)$, $U_i(x)$ are functions having a continuous derivative $U'(x) = \overline{u}(x)$, $U_i'(x) = u_i(x)$ throughout $(a, b)$ and if the series of the derivatives*

$$u(x) = u_1(x) + u_2(x) + \ldots .$$

*converges toward a continuous function $u(x)$, then the U-series can be differentiated term by term.*

For the continuous function $\overline{u}(x) - u(x)$ differs from 0 at most in a set of points $G$ satisfying Conditions $(P)$ and hence is 0 for all values of $x$.

HARVARD UNIVERSITY, CAMBRIDGE, MASSACHUSETTS, *August* 1896.

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[NOTE.—In a second paper presented to the American Mathematical Society at its Summer Meeting the geometrical method for the study of uniform convergence here set forth was treated at some length. This paper has since been printed in the Bulletin of the Society, 2d Ser., vol. III, pp. 59–86; November 1896.]

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II.—*The Integration of Series Term by Term.*